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On the dispersion relation of simply periodic waves and the corresponding spectra in the Toda lattice

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Abstract. A simply periodic wave in the nonlinear Toda lattice is discussed. The original solution by Toda needs to be amended, especially its dispersion relation. It is a special case of more general solutions that include the cases of expanded (or shrunk) lattices. The dispersion relation should be determined by prohibiting the expansion. This problem is investigated from the viewpoint of the inverse spectrum problem. A simply periodic solution, which corresponds to a single-gap spectrum, is explicitly expressed as a function of the spectrum by using the theories of algebraic functions and orthogonal polynomials. The parameters that connect the solution with the spectrum are introduced, and it is then shown that the one that was regarded as the dispersion relation is an inevitable relation among the parameters.

1. Introduction

In the late 1960s, Toda (1967a, b) discovered a nonlinear lattice that is now called the Toda lattice. It is a system of particles on a chain with nonlinear mutual interactions between adjacent particles, and rigorously bears periodic wavesolutions or solitary wavesolutions (solitons) (see also Toda (1981)). The equation of motion is written as

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} = \exp[-(y_n - y_{n-1})] - \exp[-(y_{n+1} - y_n)] \tag{1.1}$$

where $y_n = y_n(t)$ is the displacement of the particle at the nth site, $n \in \mathbb{Z}$. Equation (1.1) is often treated in the form:

$$\frac{\mathrm{d}^2 r_n}{\mathrm{d}t^2} = -\exp(-r_{n-1}) + 2\exp(-r_n) - \exp(-r_{n+1}) \tag{1.2}$$

where $r_n = r_n(t) = y_{n+1} - y_n$. Everyone recognizes this lattice as a typical example of solvable nonlinear systems.

The solution found first in the Toda lattice is the one that Toda (1967a) presented in his original paper. He showed that r_n given by

$$\exp(-r_n) = 1 + \left(\frac{K\omega}{\pi}\right)^2 \left[\mathrm{dn}^2\left(\frac{K}{\pi}(n\kappa - \omega t + \delta)\right) - \frac{E}{K}\right] \tag{1.3}$$

together with the relation

$$\omega = \pm\omega_{\mathrm{Toda}}(\kappa) \qquad \omega_{\mathrm{Toda}}(\kappa) \equiv \frac{\pi}{K}\left[\frac{1}{\mathrm{sn}^2(K\kappa/\pi)} - 1 + \frac{E}{K}\right]^{-1/2} \tag{1.4}$$

satisfies (1.2). From this fact Toda claimed a set of (1.3) and (1.4), which we hereafter refer to as the Toda solution, gives a *simply periodic* wave-like solution in his lattice, and called (1.4) the *dispersion relation* of this periodic wave.

Throughout this paper we bear it in mind that the periodicity is in both t and n. Because we are considering an infinite lattice, we must regard κ as continuous. When $\kappa/2\pi$ is irrational, strictly speaking, such a function as (1.3) is not periodic in n but *quasiperiodic*. For simplicity we hereafter regard the term 'periodic' (in n) as inclusive of the quasiperiodic case.

Without a doubt, a simply periodic solution is one of the most fundamental solutions. It seems that the Toda solution has been accepted as this. In fact, however, one must amend it, especially its dispersion relation, if it is necessary for the term 'periodic solution' to denote a periodic y_n. In section 2 we make this fact clear, and show that a relation different from (1.4) is necessary for a periodic y_n. As far as the present author is concerned, none have explicitly pointed it out, although he briefly referred to it several years ago (Yoshino 1988).

It is now well known that the Toda lattice problem can be treated in the Lax form (Flaschka 1974). Then one can obtain solutions through the inverse spectrum problem, as Kac and van Moerbeke (1975a, b) and Date and Tanaka (1976a, b) discussed (hereafter referred to as KMDT). Their results, however, do not seem to be clear enough to treat a simply periodic wave. The principal end of this paper is to obtain an expression for the solution in the form of a simple and explicit function of the spectrum, and then to discuss what the dispersion relation is in this framework, which we give in sections 3 and 4. We use the theories of algebraic functions and orthogonal polynomials. It is also widely understood that simply periodic solutions include a one-soliton solution as a limiting case. We take this limit in section 5. We conclude this paper in section 6.

2. Analysis of the Toda solution and an expression for y_n

We do not treat a set of (1.3) and (1.4) as it is. We rewrite (1.3) as

$$\exp(-r_n) = \left(\frac{K\omega}{\pi}\right)^2 \left[\frac{\mathrm{cn}^2(K\kappa/\pi)}{\mathrm{sn}^2(K\kappa/\pi)} + \mathrm{dn}^2\left(\frac{Kx_n}{\pi}\right)\right] \tag{2.1}$$
$$x_n = x_n(t) \equiv n\kappa - \omega t + \delta.$$

A set of (2.1) and (1.4) is evidently equivalent to the Toda solution. In the following we show that equation (2.1) solely satisfies (1.2) irrespective of ω or κ.

We rewrite (2.1) in terms of the elliptic theta functions which are often more tractable than the Jacobian elliptic functions. There are several kinds of notations for the theta functions. In this paper we adopt the notation used in Whittaker and Watson (1927) (hereafter referred to as WW), i.e. they are written as $\vartheta_j(z, q)$ ($\equiv \vartheta_j(z|\tau)$; $j = 1, 2, 3, 4$; $q = \mathrm{e}^{\mathrm{i}\pi\tau}$), and the zeros of $\vartheta_1(z|\tau)$ are at $z = \pi(l+m\tau)$ ($l, m \in \mathbb{Z}$), etc. For brevity we omit the second argument, q (called nome) or τ, except in section 5, as we have been omitting the modulus (usually referred to as k) from the Jacobian functions and the complete elliptic integrals K and E.

Equation (2.1) then becomes

$$\exp(-r_n) = \tfrac{1}{4}\vartheta_3^2\vartheta_4^2\omega^2 \left[\frac{\vartheta_2^2(\frac{1}{2}\kappa)}{\vartheta_1^2(\frac{1}{2}\kappa)} + \frac{\vartheta_3^2(\frac{1}{2}x_n)}{\vartheta_4^2(\frac{1}{2}x_n)}\right] \tag{2.2}$$

$$= \left[\frac{\vartheta_1'\omega}{2\vartheta_1(\frac{1}{2}\kappa)}\right]^2 \frac{\vartheta_4(\frac{1}{2}x_{n-1})\vartheta_4(\frac{1}{2}x_{n+1})}{\vartheta_4^2(\frac{1}{2}x_n)}. \tag{2.3}$$

We have reduced (2.2) to (2.3) by using the identity (see WW, p 488):

$$\vartheta_2^2\vartheta_4(z+a)\vartheta_4(z-a) = \vartheta_1^2(a)\vartheta_3^2(z) + \vartheta_2^2(a)\vartheta_4^2(z).$$

Then we readily observe that equation (1.2) holds in such a manner as

$$\text{l.h.s.} = \tfrac{1}{4}\vartheta_3^2\vartheta_4^2\omega^2\left[-\frac{\vartheta_3^2(\frac{1}{2}x_{n-1})}{\vartheta_4^2(\frac{1}{2}x_{n-1})} + 2\frac{\vartheta_3^2(\frac{1}{2}x_n)}{\vartheta_3^2(\frac{1}{2}x_n)} - \frac{\vartheta_3^2(\frac{1}{2}x_{n+1})}{\vartheta_4^2(\frac{1}{2}x_{n+1})}\right] = \text{r.h.s.}$$

The reduction of the r.h.s. is direct from (2.2). The l.h.s. is reduced by using (2.3) and the identity†:

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\vartheta_4(z) = \vartheta_3^2\vartheta_4^2\frac{\vartheta_3^2(z)}{\vartheta_4(z)} + \frac{\vartheta_2''}{\vartheta_2}. \tag{2.4}$$

We have thus shown that equation (2.1) solely satisfies (1.2).

We should impose periodicity on y_n as mentioned before. We thus need to obtain an expression for y_n. It is easy to express $y_n - y_0$ from (2.3). However, we intend to express y_n without any reference to other displacements. Let

$$f_n = f_n(t) = y_n + \ln\left[\frac{\vartheta_4(\frac{1}{2}x_n)}{\vartheta_4(\frac{1}{2}x_{n-1})}\right]$$

then we obtain

$$f_{n+1} = f_n + 2\ln\left|\frac{2\vartheta_1(\frac{1}{2}\kappa)}{\vartheta_1'\omega}\right| \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}f_n = 0. \tag{2.5}$$

The former of (2.5) is direct from (2.3), and the latter is from (1.1), (2.2) and (2.4). We thus readily obtain from (2.5)

$$y_n = A + Vt + 2n\ln\left|\frac{2\vartheta_1(\frac{1}{2}\kappa)}{\vartheta_1'\omega}\right| + \ln\left[\frac{\vartheta_4(\frac{1}{2}x_{n-1})}{\vartheta_4(\frac{1}{2}x_n)}\right] \tag{2.6}$$

where A and V are arbitrary (integration) constants dependent on neither t nor n.

Equation (2.6) does not always give a periodic y_n. In order for y_n to be periodic in both t and n, two conditions: $V = 0$, and

$$\omega = \pm\omega_{\text{peri}}(\kappa) \qquad \omega_{\text{peri}}(\kappa) \equiv \frac{2\vartheta_1(\frac{1}{2}\kappa)}{\vartheta_1'} \tag{2.7}$$

are necessary (and sufficient). We can readily make the former hold because V is arbitrary. The latter states that ω is no longer arbitrary but must be a function of κ (and q). This should be interpreted as the dispersion relation for the simply periodic wave.

When $|\omega|$ differs from $\omega_{\text{peri}}(\kappa)$, then y_n is not periodic in n, and the lattice exhibits uniform expansion (when $|\omega| < \omega_{\text{peri}}(\kappa)$) or shrinkage (when $|\omega| > \omega_{\text{peri}}(\kappa)$) if we regard y_n as longitudinal. The dispersion relation (2.7) is thus equivalent to the condition that the lattice should not expand or shrink.

We see that $\omega_{\text{Toda}}(\kappa)$ differs from $\omega_{\text{peri}}(\kappa)$, because the latter is an entire function as a function of $\kappa \in \mathbb{C}$ while the former is not. Namely, the Toda solution does not give a periodic y_n. We note $\omega_{\text{Toda}}(\kappa) < \omega_{\text{peri}}(\kappa)$ everywhere in $0 < \kappa < \pi$, where the inequality may become the equality in some limiting cases, for example $q \to 0$.

Finally in this section we make three remarks on solution (2.6).

† This identity is rarely referred to in this form. It is usually found in the form: $\mathrm{dn}^2 u = (\mathrm{d}^2/\mathrm{d}u^2)\ln\Theta(u) + E/K$, where $\Theta(u)$ is the Jacobian theta function: $\Theta(u) = \vartheta_4((\pi/2K)u)$ (see WW, pp 479 and 518).

(i) In (2.6) there are four arbitrary parameters unless the periodicity is imposed on y_n. They are V, ω, κ and q. We do not take two constants, A and δ, into account, because they merely play the roles of fixing the origins of y_n and t, respectively. When we impose the periodicity, two of the four become fixed and the other two (i.e. κ and q) remain arbitrary.

(ii) Because the differentiation with respect to t in (1.1) or (1.2) is of the second order only, both positive and negative signs are allowed for ω. So we needed to write the double sign ($\pm$) in (2.7). We can, however, regard ω as positive definite based on the following consideration. Evidently, we can take κ in $0 < \kappa < 2\pi$. We can then rewrite the theta functions in (2.6) as

$$\vartheta_4(\tfrac{1}{2}[n\kappa-(-\omega)t+\delta]) = \vartheta_4(\tfrac{1}{2}(n\kappa'-\omega t-\delta)) \qquad \kappa' = 2\pi - \kappa$$

which implies that a pair of $-\omega$ (< 0) and κ is equivalent to a pair of ω (> 0) and $2\pi - \kappa$. We can thus remove the double sign from (2.7).

(iii) In the last (most significant) term in (2.6), the theta functions appear in the form of a quotient. It might be useful if one could write it in terms of the Jacobian elliptic functions. However, one cannot do so. The reason is that the theta quotient is not a doubly periodic function. Let $\vartheta_4(z-\frac{1}{2}\kappa)/\vartheta_4(z) \equiv \chi(z)$ and regard it as a function of z ($\equiv \frac{1}{2}x_n$) $\in \mathbb{C}$. Then $\chi(z+\pi) = \chi(z)$ is trivial, and $\chi(z+\pi\tau) = \exp(\frac{1}{2}\mathrm{i}\kappa)\chi(z)$ (WW, p 463). So we cannot find the second periodicity when $\kappa/2\pi$ is irrational, although $\chi(z)$ becomes doubly periodic with periods π and $2m\pi\tau$ (or $m\pi\tau$ if l is even) when $\kappa/2\pi$ is accidentally rational ($= l/m$).

3. Preliminary considerations for the inverse problem

In this paper we define the Lax-form variables in a slightly different manner from Flaschka (1974) or KMDT. We let

$$a_n = a_n(t) = \exp[\tfrac{1}{2}(y_n - y_{n+1})] \qquad b_n = b_n(t) = \frac{\mathrm{d}y_n}{\mathrm{d}t} \tag{3.1}$$

and introduce two tridiagonal matrices, $\mathbf{L} = \mathbf{L}(t)$ (symmetric) and $\mathbf{B} = \mathbf{B}(t)$ (antisymmetric), whose elements at (i, j) are ($n \in \mathbb{Z}$)

$$L_{i,j} = \begin{cases} b_n & (i,j) = (n,n) \\ a_n & (i,j) = (n,n+1) \text{ or } (n+1,n) \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

$$B_{i,j} = \begin{cases} \frac{1}{2}a_n & (i,j) = (n+1,n) \\ -\frac{1}{2}a_n & (i,j) = (n,n+1) \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

These matrices must be *infinite*-dimensional because we are also concerned with quasiperiodic cases. The equations of motion for a_n, b_n and $\mathbf{L}$ become

$$\frac{\mathrm{d}a_n}{\mathrm{d}t} = \tfrac{1}{2}a_n(b_n - b_{n+1}) \qquad \frac{\mathrm{d}b_n}{\mathrm{d}t} = a_{n-1}^2 - a_n^2 \tag{3.4}$$

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{BL} - \mathbf{LB}. \tag{3.5}$$

It then follows from (3.5) (Flaschka 1974) that $\mathbf{L}(t)$ is unitarily equivalent to $\mathbf{L}(0)$, i.e. the spectrum of $\mathbf{L}(t)$, say F, is conserved as t. Therefore we can construct $\mathbf{L}(t)$ (and $y_n(t)$ through $a_n(t)$) from thus conserved F by way of the inverse problem (see also van

Moerbeke (1976)). We should note its dependence on n that we can obtain from a solution of the inverse problem. We need to use (3.4) or (3.5) to determine its dependence on t.

The simply periodic solution corresponds to a band spectrum composed of two subbands separated by a gap:

$$F = [\lambda_4, \lambda_3] + [\lambda_2, \lambda_1] \qquad \lambda_4 < \lambda_3 < \lambda_2 < \lambda_1.$$

The solution in this framework was given by KMDT, but it does not seem to be clear. One reason is, we think, that they did not give an expression for y_n (nor a_n) in the form of an explicit function of F. In this paper we do so.

3.1. Qualitative and quantitative parameters

We classify the spectra from the following elementary consideration. We linearly transform $\mathbf{L}$ as

$$\mathbf{L} \to \mathbf{L}' = c\mathbf{L} + d\mathbf{I} \qquad (c > 0) \qquad \text{i.e. } \{a_n, b_n\} \to \{a_n', b_n'\} = \{ca_n, cb_n + d\}$$

then evidently $F \to F' = \{\lambda' = c\lambda + d | \lambda \in F\}$. In this section F', λ', c and d are read in this manner. It is natural to regard the new (primed) system as being substantially equivalent to the original system. We thus introduce a class of the spectra by this equivalency. Hereafter we use the symbol Γ to denote the class:

$$F \in \Gamma = \{F' | c, d \in \mathbb{R}; c > 0\}.$$

Briefly speaking, the class is composed of spectra whose configurations are proportional to one another.

It is obvious that F is characterized by four scalars, i.e. by λ_m ($m = 1, 2, 3, 4$). The above classification strongly suggests that we should rather take two kinds of parameters to characterize F; those that play the roles of identifying a class to which F belongs, and those that identify F in the class. We call the former *qualitative* and the latter *quantitative*. In other words, qualitative parameters, which must be class functions of Γ, determine the relative structure of a spectrum, and quantitative parameters are, as c and d in the above, the scaling and the translating factors of the spectrum or the corresponding matrix. Any λ_m is, by itself, neither qualitative nor quantitative. In the next section we shall use parameters, each of which is of either kind. Evidently, two quantitative (scalar) parameters are necessary and sufficient, as which we shall choose a pair that behaves the same as a pair of c and d. Then exactly two qualitative parameters are necessary and sufficient.

3.2. Algebraic functions and canonical basis

According to KMDT, algebraic functions associated with $[R(\lambda; F)]^{1/2}$ are essential, where

$$R(\lambda; F) \equiv (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4).$$

The corresponding Riemann surface, say $\mathcal{S}$, is of genus unity. It consists of two replicas of the Riemann sphere, which are both cut along $[\lambda_4, \lambda_3]$ and $[\lambda_2, \lambda_1]$, and glued to each other at the cut edges. We denote one replica by $\mathbb{C}^{\mathrm{p}}$ (which is often called the physical sheet) and the other by $\mathbb{C}^{\mathrm{u}}$ (the unphysical sheet). Each replica contains the infinity point, which we denote by ∞ on $\mathbb{C}^{\mathrm{p}}$ and by ∞^* on $\mathbb{C}^{\mathrm{u}}$. We make symbols λ, λ_1, etc, denote not only a point on $\mathcal{S}$ but also its value (its projection on $\mathbb{C}$). We hereafter write $[R(\lambda; F)]^{1/2}$ meaning a branch such that $[R(\lambda; F)]^{1/2}/\lambda^2 \to 1$ (or -1) for $\lambda \to \infty$ (or ∞^*).

According to the convention (Siegel 1969, p 49) we canonically dissect $\mathcal{S}$ as is shown in figure 1(a). We then obtain a simply connected cut surface $\mathcal{S}^{\mathrm{c}}$ surrounded by the oriented

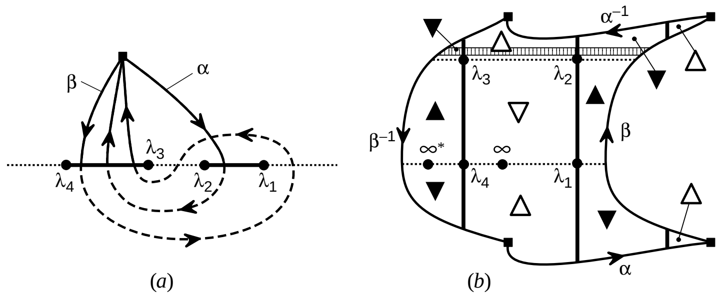

Figure 1. The Riemann surface $\mathcal{S}$ is canonically dissected as in (a). The full and the broken curves denote they are in $\mathbb{C}^{\mathrm{p}}$ and $\mathbb{C}^{\mathrm{u}}$, respectively. Then the dissected surface $\mathcal{S}^{\mathrm{c}}$ looks like (b). In (b) we show by four kinds of triangles which half (of $\mathbb{C}^{\mathrm{p}}$ or $\mathbb{C}^{\mathrm{u}}$) the area is; △ and ▲ denote the upper halves of $\mathbb{C}^{\mathrm{p}}$ and $\mathbb{C}^{\mathrm{u}}$, respectively, and ▽ and ▼ mean the lower halves of $\mathbb{C}^{\mathrm{p}}$ and $\mathbb{C}^{\mathrm{u}}$, respectively. If we cut $\mathcal{S}$ along the zebra strap shown in (b) (without cutting along α or β) then we obtain $\mathcal{S}^*$.

boundary $\partial\mathcal{S}^{\mathrm{c}}$, which is generated by a pair of crosscuts α and β: $\partial\mathcal{S}^{\mathrm{c}} = \alpha\beta\alpha^{-1}\beta^{-1}$. Figure 1($b$) schematically shows where the upper and the lower halves of each replica are located (and how they are dissected) in $\mathcal{S}^{\mathrm{c}}$.

By the canonical dissection we introduce the canonical basis of the first-kind Abelian differential: $\mathrm{d}u\,(\lambda; F) \propto \mathrm{d}\lambda\,/[R(\lambda; F)]^{1/2}$, which is normalized by the α-period and defines a constant τ by the β-period:

$$u(\alpha; F) \equiv \int_\alpha \mathrm{d}u\,(\lambda; F) = \pi \qquad u(\beta; F) \equiv \int_\beta \mathrm{d}u\,(\lambda; F) = \pi\tau. \tag{3.6}$$

As is well known, τ (or the corresponding q) becomes the second argument of the relevant theta functions. We readily see that τ is pure imaginary with a positive imaginary part and is qualitative: $\tau = \tau(\Gamma)$, which implies $q \in \mathbb{R}$ and $q = q(\Gamma)$.

We define the integral of $\mathrm{d}u\,(\lambda; F)$ by

$$u(\lambda; F) = \int_{\lambda_4}^{\lambda} \mathrm{d}u\,(\lambda; F)$$

which bears multivalency on $\mathcal{S}$ caused by (3.6). We have located the origin of the integration at λ_4 merely for simplicity. Figure 2 shows how $u(\lambda; F)$ varies when λ moves in $\mathcal{S}^{\mathrm{c}}$.

3.3. *Orthogonal polynomials and the exterior mapping function*

The present problem is closely related through $\mathbf{L}$ to the theory of orthogonal polynomials† defined by a weight whose support is F, although a semi-infinite $\mathbf{L}$ is treated in the latter (Magnus 1979, Turchi *et al* 1982, Peherstorfer 1991).

The orthogonal polynomials on F, especially their asymptotic behaviours, are well described in terms of an exterior mapping function of F (Yoshino 1987), which we denote by $\Phi(\lambda; F)$. It is an analytic function defined uniquely by the following, accompanied with the introduction of three definite functions of F. We refer to the three functions as $\gamma(F)$, $\Lambda(F)$ and $\kappa(F)$. The function $\Phi(\lambda; F)$ behaves around ∞ ($\in \mathbb{C}^{\mathrm{p}}$) as

$$\Phi(\lambda; F) = \lambda - \Lambda(F) + \mathrm{O}(\lambda^{-1}) \tag{3.7}$$

† In the theory of orthogonal polynomials it is a convention that diagonal elements are referred to as a_n and off-diagonal elements are refered to as b_n, unlike the Toda lattice problem.

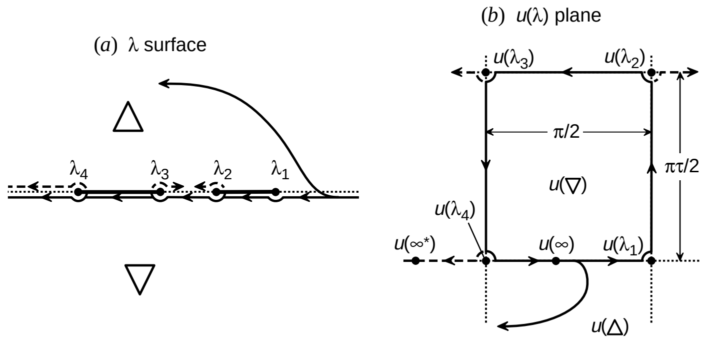

Figure 2. The behaviours of $u(\lambda; F)$ when λ mainly moves along the real axis. The full and the broken curves denote that they are in $\mathbb{C}^{\mathrm{p}}$ and $\mathbb{C}^{\mathrm{u}}$, respectively. $\triangle$ and $\bigtriangledown$ mean the upper half and the lower half, respectively, of $\mathbb{C}^{\mathrm{p}}$.

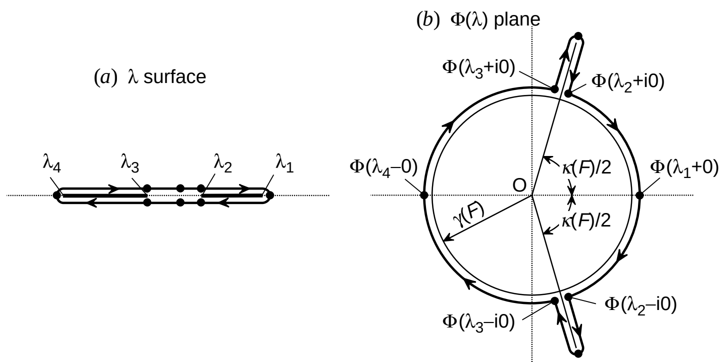

Figure 3. Schematic depiction of the exterior mapping function: $\Phi(\lambda; F)$. (a) shows the physical sheet. The second argument F is omitted in (b) for brevity.

(by which $\Lambda(F)$ is defined), and maps the exterior of $[\lambda_4, \lambda_1]$ in $\mathbb{C}^{\mathrm{p}}$ conformally onto the exterior of a disk, with two whiskers issuing, centred at the origin as is shown in figure 3. The function $\kappa(F)$ is defined as twice the argument of the upper whisker in figure 3(b), and $0 < \kappa(F) < 2\pi$. We shall later (in section 4.1) see that $\kappa(F)$ equals the wavenumber of the periodic wave. The function $\gamma(F)$ (> 0) is defined as the radius of the disk. It equals a quantity that appears in several situations in mathematics, and thus has several names according to the situations: transfinite diameter, logarithmic capacity and Tchebycheff constant (Hille 1962).

By definition we find $\Phi(\lambda'; F') = c\Phi(\lambda; F) = \lambda' - d - c\Lambda(F) + \cdots$, which gives rise to the relations:

$$\gamma(F') = c\gamma(F) \qquad \Lambda(F') = c\Lambda(F) + d. \tag{3.8}$$

Equations (3.8) imply that we can choose $\gamma(F)$ and $\Lambda(F)$ as a pair of quantitative parameters, and hereafter we do so.

There is a more confirmative reason why we adopt this pair. From the analysis of asymptotic behaviours of orthogonal polynomials we obtain the limiting relations (Yoshino 1987):

$$\lim_{n\to\infty}\left(\prod_{j=0}^{n-1} a_j(t)\right)^{1/n} = \gamma(F) \qquad \lim_{n\to\infty}\left(\frac{1}{n}\sum_{j=0}^{n-1} b_j(t)\right) = \Lambda(F). \tag{3.9}$$

These relations, which give rise to (3.8) again, definitely show that $\gamma(F)$ and $\Lambda(F)$ are measures of the size (width) and the centre (in some sense) of F, respectively. In terms of the Toda lattice problem, they are directly related to a uniform expansion (or shrinkage) and a velocity of a translational motion, respectively (see definitions (3.1)). Namely, this pair bears substantial connections with both F and y_n.

3.4. Multivalency of the exterior mapping function

We note that $\Phi(\lambda; F)$, which was defined in $\mathbb{C}^{\mathrm{p}}$, can be analytically continued to $\mathbb{C}^{\mathrm{u}}$. Nevertheless it is not algebraic nor univalent on $\mathcal{S}$. In order to understand the multivalency it is useful to examine its logarithmic differential: $\mathrm{d}\phi\,(\lambda; F)$, $\phi(\lambda; F) \equiv \ln \Phi(\lambda, F)$. It has the form of (Yoshino 1987)

$$\mathrm{d}\phi\,(\lambda; F) = \frac{\lambda - s}{[R(\lambda; F)]^{1/2}}\mathrm{d}\lambda \qquad (s \in (\lambda_3, \lambda_2) \subset \mathbb{R}) \tag{3.10}$$

i.e. it is an Abelian differential of the third kind on $\mathcal{S}$. The poles are at ∞ and ∞^* with the residues -1 and 1 respectively. A contour integral that encloses the pole or poles thus induces the additive multivalency of $\phi(\lambda; F)$ by an integral times $2\pi\mathrm{i}$, which does not make $\Phi(\lambda; F)$ multivalent. In view of the α-period and the β-period, we obtain from the properties shown in figure 3

$$\phi(\alpha; F) \equiv \int_\alpha \mathrm{d}\phi\,(\lambda; F) = 0 \qquad \phi(\beta; F) \equiv \int_\beta \mathrm{d}\phi\,(\lambda; F) = -\mathrm{i}\kappa(F). \tag{3.11}$$

Namely, the multivalency of $\Phi(\lambda; F)$ results from this β-period only.

We see that the β-period is brought about by a round trip of λ (on $\mathcal{S}$) in the vertical direction in figure 1(b). We can thus avoid the multivalency of $\Phi(\lambda; F)$ if we prohibit λ from making such a round trip. We hence cut $\mathcal{S}$ along the zebra strap shown in figure 1(b), and hereafter regard λ as moving in this cut surface, which we denote by $\mathcal{S}^*$. It is topologically equivalent to a tube with both ends open or a balloon with two holes. Then $\Phi(\lambda; F)$ becomes univalent. Any point, such as ∞, λ_1, etc, is also regarded as being located in $\mathcal{S}^*$. The strap, which is closed on $\mathcal{S}$, runs just above the dotted line: $\lambda_3 - \lambda_2 - \lambda_3$ in figure 1(b). The imaginary part of $u(\lambda; F)$ hence becomes definite for any $\lambda \in \mathcal{S}^*$, and ranges as (see figure 2(b))

$$-\tfrac{1}{2}\pi(\tau/\mathrm{i}) < \Im u(\lambda; F) \leqslant \tfrac{1}{2}\pi(\tau/\mathrm{i}).$$

We should regard the first inequality as inclusive of the equality in some limiting cases. For example, we should write $u(\lambda_3 \pm \mathrm{i}0; F) = \mp\frac{1}{2}\pi\tau \pmod{\pi}$ (note $\lambda_3 \equiv \lambda_3 - \mathrm{i}0$) in $\mathbb{C}^{\mathrm{p}}$, and the sign becomes opposite in $\mathbb{C}^{\mathrm{u}}$. We need to bear in mind that $u(\lambda; F)$ still has the ambiguity in mod π on $\mathcal{S}^*$.

3.5. Estimates for $\omega(F)$ and $\kappa(F)$

According to KMDT, the angular frequency $\omega(F)$ and the wavenumber $\kappa(F)$ of the simply periodic wave are related to $\mathrm{d}u\,(\lambda; F)$ and $u(\lambda; F)$ by

$$\mathrm{d}u\,(\lambda; F) = \frac{-\frac{1}{2}\omega(F)}{[R(\lambda; F)]^{1/2}}\,\mathrm{d}\lambda \tag{3.12}$$

$$u(\infty; F) = -u(\infty^*; F) = \tfrac{1}{4}\kappa(F). \tag{3.13}$$

In this paper we regard $\omega(F)$ as a constant defined by (3.12). In view of $\kappa(F)$, in contrast, we have already defined it (in section 3.3) by the argument of $\Phi(\lambda_2 \pm \mathrm{i}0)$ or $\Phi(\lambda_3 \pm \mathrm{i}0)$ (see figure 3(*b*)), or equivalently by the latter of (3.11). We see that $\kappa(F)$ given by (3.13) is consistent with our definition, because integration of $u(\lambda; F)\mathrm{d}\phi(\lambda; F)$ over $\partial\mathcal{S}^{\mathrm{c}}$ becomes

$$-u(\beta; F)\phi(\alpha; F) + u(\alpha; F)\phi(\beta; F) = 2\pi\mathrm{i}[u(\infty^*; F) - u(\infty; F)]$$

(see (3.6) and (3.11)), where the residues of $\mathrm{d}\phi(\lambda; F)$ are used on the r.h.s. Therefore we can regard $\kappa(F)$ as being defined by (3.13).

Equation (3.12) gives rise to $\mathrm{d}u(\lambda'; F')/\omega(F') = c^{-1}\mathrm{d}u(\lambda; F)/\omega(F)$ because $R(\lambda'; F') = c^4 R(\lambda; F)$. The normalization of $\mathrm{d}u(\lambda; F)$ thus implies $\omega(F') = c\omega(F)$, i.e. $\omega(F)/\gamma(F)$ is qualitative while $\omega(F)$ itself is not. It also implies, directly from (3.13), that $\kappa(F)$ is qualitative by itself: $\kappa(F) = \kappa(\Gamma)$.

4. Solution through the inverse problem

We are now ready to obtain a solution of the inverse problem. We follow a procedure based, to a large extent, on algebraic function theory (see Siegel (1971)). Namely, we shall express an algebraic function (of λ on $\mathcal{S}$), $\Phi(\lambda; F)$ or a product of them, any of which is univalent in $\mathcal{S}^*$, in terms of the theta functions whose arguments contain $u(\lambda)$. We contract $u(\lambda; F)$ as $u(\lambda)$ for brevity. The second argument τ of the theta functions is the one given by (3.6), which we do not explicitly specify.

We first emphasize the following. In our derivation, $\omega(F)$ and $\kappa(F)$ are constants (independent of λ) defined by (3.12) and (3.13), respectively. We shall then show that the angular frequency and the wavenumber are given by them. Hereafter we often refer to $\kappa(F)$ as κ for brevity.

4.1. *Expression for a_n as a function of the spectrum*

We derive an expression for a_n by taking three steps. We first suppose that F generates a (quasi)periodic sequence of $\{a_n, b_n\}$, and then introduce a function:

$$g_n(\lambda; F) = \frac{1}{\lambda - b_n - a_n^2/\{\lambda - b_{n+1} - a_{n+1}^2/[\lambda - b_{n+2} - a_{n+2}^2/(\lambda - \cdots)]\}} \tag{4.1}$$

$$= \frac{1}{\lambda - b_n - a_n^2 g_{n+1}(\lambda; F)} \tag{4.2}$$

$$= \lambda^{-1} + b_n\lambda^{-2} + \mathrm{O}(\lambda^{-3}) \qquad \text{as } \lambda \to \infty. \tag{4.3}$$

We define $g_n(\lambda; F)$ by the continued Jacobi fraction of (4.1) for $\lambda \in \mathbb{C}^{\mathrm{p}}$ at a sufficient distance from F, and then analytically continue it to $\mathcal{S}$, which makes it algebraic on $\mathcal{S}$ (Magnus 1979). It has two simple zeros, one of which is ∞ as equation (4.3) shows, and two simple poles, one of which is ∞^*. The other zero, say ζ_n, is located in $[\lambda_3, \lambda_2]$ on either $\mathbb{C}^{\mathrm{p}}$ or $\mathbb{C}^{\mathrm{u}}$ according to the occasion, and the case is the same with the other pole, say π_n. Namely, they are both located at one open end of the tube to which $\mathcal{S}^*$ is equivalent, or just below the zebra strap in figure 1(*b*). We thus obtain (see also figure 2(*b*))

$$\Im u(\zeta_n) = \Im u(\pi_n) = \tfrac{1}{2}\pi(\tau/\mathrm{i}).$$

We readily find $\zeta_n = \pi_{n+1}$ because of the recursion relation (4.2). We also have

$$u(\zeta_n) + u(\infty) = u(\pi_n) + u(\infty^*) \qquad (\mathrm{mod}\ \pi)$$

by applying the theorem of Abel (Siegel 1971, p 144) to $g_n(\lambda; F)$. Therefore we obtain

$$u(\zeta_n) = u(\pi_{n+1}) = u(\zeta_0) - \tfrac{1}{2}n\kappa \qquad (\mathrm{mod}\ \pi) \tag{4.4}$$

i.e. $\{u(\zeta_n)\}$ and $\{u(\pi_n)\}$ form arithmetic sequences.

The fact that ∞^* is a pole of $g_n(\lambda; F)$ implies (see (4.2) again) $\lambda - b_n - a_n^2 g_{n+1}(\lambda; F) = \mathrm{O}(\lambda^{-1})$ as $\lambda \to \infty^*$, i.e.

$$g_n(\lambda; F) = (a_{n-1}^2)^{-1}\lambda + \mathrm{O}(\lambda^0) \qquad \text{as } \lambda \to \infty^* \tag{4.5}$$

which, we note, contains a_{n-1} despite the fact that $g_n(\lambda; F)$ was defined without a_{n-1} in (4.1). This is never an inconsistency because all elements of $\{a_n, b_n\}$ are related to one another through F.

Next we examine $\Phi(\lambda; F)$. Let us trace $\phi(\lambda; F)$ for λ moving as $\infty \to \lambda_4 \to \infty^*$ (along the segment in figure 1(b)) with (3.7), (3.10) and $\Phi(\lambda_4; F) = -\gamma(F)$ (see figure 3) taken into account. Then we find

$$\Phi(\lambda; F) = \gamma(F)^2\lambda^{-1} + \mathrm{O}(\lambda^{-2}) \qquad \text{as } \lambda \to \infty^*. \tag{4.6}$$

We can thus rewrite (3.7), (4.3), (4.5) and (4.6) in a form:

$$g_n(\lambda; F)\Phi(\lambda; F) = \begin{cases} 1 + \mathrm{O}(\lambda^{-1}) & \text{as } \lambda \to \infty \\ [\gamma(F)/a_{n-1}]^2 + \mathrm{O}(\lambda^{-1}) & \text{as } \lambda \to \infty^*. \end{cases} \tag{4.7}$$

Although it is not algebraic, $\Phi(\lambda; F)$ can be expressed in terms of the theta functions because its logarithmic differential is of the third kind with integral residues. We then obtain†

$$\Phi(\lambda; F) = \gamma(F)\frac{\vartheta_1(u(\lambda) - u(\infty^*))}{\vartheta_1(u(\lambda) - u(\infty))} \tag{4.8}$$

by using the residues (-1 at ∞ and 1 at ∞^* as given in section 3.3) and $\Phi(\lambda_4; F) = -\gamma(F)$. Note the ambiguity of $u(\lambda)$ may change the sign of $\vartheta_1(u(\lambda) - \cdots)$, but in any case the r.h.s. of (4.8) is kept invariant.

Equation (4.8) shows again that the arguments of the whiskers in figure 3(b) equal $\pm\frac{1}{2}\kappa(F)$, for we obtain, for example

$$\vartheta_1(u(\lambda_3 \pm \mathrm{i}0) - u(\infty)) = \vartheta_1(\mp\tfrac{1}{2}\pi\tau - \tfrac{1}{4}\kappa) = \mp\mathrm{i}\exp(-\tfrac{1}{4}\mathrm{i}\pi\tau \mp \tfrac{1}{4}\mathrm{i}\kappa)\vartheta_4(\tfrac{1}{4}\kappa)$$

from the second half period ($\frac{1}{2}\pi\tau$) relations (WW, p 464) (the signs of the second and the third expressions can become opposite due to the ambiguity of $u(\lambda_3 \pm \mathrm{i}0)$ in mod π). Hereafter we shall often apply the second half period relations without notice.

Bearing (4.7) in mind, we finally consider a function: $g_n(\lambda; F)\Phi(\lambda; F)$. Its logarithmic differential is of the third kind with poles at ζ_n and π_n whose residues are 1 and -1 respectively. We hence obtain

$$\begin{aligned} g_n(\lambda; F)\Phi(\lambda; F) &= \frac{\vartheta_1(u(\infty) - u(\pi_n))\vartheta_1(u(\lambda) - u(\zeta_n))}{\vartheta_1(u(\infty) - u(\zeta_n))\vartheta_1(u(\lambda) - u(\pi_n))} \\ &= \frac{\vartheta_4(\frac{1}{2}(n-1)\kappa - \frac{1}{2}\sigma)\vartheta_4(u(\lambda) - u(\infty) + \frac{1}{2}n\kappa - \frac{1}{2}\sigma)}{\vartheta_4(\frac{1}{2}n\kappa - \frac{1}{2}\sigma)\vartheta_4(u(\lambda) - u(\infty) + \frac{1}{2}(n-1)\kappa - \frac{1}{2}\sigma)} \end{aligned} \tag{4.9}$$

where we have used the value at ∞ (see (4.7)), substituted (4.4), and let

$$u(\zeta_0) = u(\infty) + \tfrac{1}{2}\pi\tau + \tfrac{1}{2}\sigma \qquad (\sigma \in \mathbb{R}). \tag{4.10}$$

† Strictly speaking, we must settle a factor of $\exp[Cu(\lambda)]$ ($C \in \mathbb{C}$) by which the r.h.s. of (4.8) can be multiplied. We obtain $C = 0$ by the univalency of $\Phi(\lambda; F)$ in $\mathcal{S}^*$ and the fact that $\Phi(\lambda; F) \in \mathbb{R}$ for $\lambda \in (-\infty, \lambda_4]$. Similar considerations are necessary in other third-kind differential cases.

We then obtain by using the value at ∞^* (see (4.7) again)

$$a_n = \gamma(F)\frac{[\vartheta_4(\frac{1}{2}v_{n-1})\vartheta_4(\frac{1}{2}v_{n+1})]^{1/2}}{\vartheta_4(\frac{1}{2}v_n)} \tag{4.11}$$

$$v_n = v_n(t; F) \equiv n\kappa(F) - \sigma. \tag{4.12}$$

We have thus found that the wavenumber equals $\kappa(F)$ (defined by (3.13)), as expected in section 3.5. We note that equation (4.11) has a form that explicitly reflects the first of relations (3.9). We do not write an expression for b_n, because it is less simple than (4.11) and seems to be less useful to derive an expression for y_n (see (3.1)).

Among the quantities that have appeared in this section, we see that $\gamma(F)$, $\kappa(\Gamma)$, $\tau(\Gamma)$ and the functional forms of $u(\cdot; F)$, $\Phi(\cdot; F)$ and $\vartheta_j(\cdot|\tau)$ are independent of t, because they depend only on F that is conserved as t. Variables that depend on t are a_n, ζ_n, π_n, v_n, σ and (the functional form of) $g_n(\cdot; F)$. It thus suffices to examine the dependence of σ, for we can relate any other variable to σ through (4.10) or (4.12).

4.2. Dependence on time

We determine the dependence of σ on t. For this purpose we derive an equation of motion for $g_n(\lambda; F)$. We rewrite the eigenvalue equation for $\mathbf{L}(t)$, i.e. $\mathbf{L}(t)\boldsymbol{\psi}(\lambda, t) = \lambda\boldsymbol{\psi}(\lambda, t)$, into a form (see (3.2)):

$$a_{n-1}(t)\psi_{n-1}(\lambda, t) + b_n(t)\psi_n(\lambda, t) + a_n(t)\psi_{n+1}(\lambda, t) = \lambda\psi_n(\lambda, t) \tag{4.13}$$

$\psi_n(\lambda, t)$ being the nth element of $\boldsymbol{\psi}(\lambda, t)$. We see that λ is independent of t because of the unitary equivalency between $\mathbf{L}(t)$ and $\mathbf{L}(0)$, and then regard it as a variable in $\mathcal{S}^*$, i.e. we treat it in the same manner as in the preceding sections. We must bear in mind that there are two independent variables, λ and t, in the following argument. We thus write $g_n(\lambda, t)$ for $g_n(\lambda; F)$. We denote the (partial) differentiation with respect to t by ∂_t.

Equation (4.13) has two linearly independent (concerning n) solutions for any λ (with t fixed), because it is a three-term recursion relation. We choose the solutions, say $\boldsymbol{\psi}^+(\lambda, t)$ and $\boldsymbol{\psi}^-(\lambda, t)$, such that

$$\psi_n^{\pm}(\lambda, t) \propto \lambda^{\pm n} \qquad \text{as } \lambda \to \infty\ (\in \mathbb{C}^{\mathrm{P}})$$

for $n > 0$, i.e. ∞ is an n-fold pole for ψ_n^+ or an n-fold zero for ψ_n^-. We easily obtain $\boldsymbol{\psi}^+$ by letting $\psi_{-1}^+(\lambda, t) = 0$ and $\psi_0^+(\lambda, t) = 1$, and using (4.13) one after another (as n=0, 1, 2, ...), which evidently generates $\boldsymbol{\psi}^+$ such that $\psi_n^+(\lambda, t)$ is an n-degree polynomial of λ. We construct $\boldsymbol{\psi}^-$ recursively by†

$$\psi_0^-(\lambda, t) = 1 \qquad \psi_n^-(\lambda, t) = a_{n-1}(t)g_n(\lambda, t)\psi_{n-1}^-(\lambda, t). \tag{4.14}$$

This has the desired behaviour at $\lambda \to \infty$ as equation (4.3) directly shows, and makes (4.13) hold by virtue of (4.2). We can have $\psi_n^{\pm}$ for $n < 0$ in a similar manner as for $n > 0$, i.e. by applying (4.13) and (4.14) reversely, but they are not necessary in the following argument.

It is shown by using (3.5) (Toda 1981) that each $\partial_t\boldsymbol{\psi}^{\pm}(\lambda, t) - \mathbf{B}(t)\boldsymbol{\psi}^{\pm}(\lambda, t)$ is also a solution of (4.13) (with the same λ as $\boldsymbol{\psi}^{\pm}(\lambda, t)$), i.e. it is expressed by a linear combination of $\boldsymbol{\psi}^+(\lambda, t)$ and $\boldsymbol{\psi}^-(\lambda, t)$, the coefficients being functions of λ and t (but independent of n). Let us take $\partial_t\boldsymbol{\psi}^- - \mathbf{B}\boldsymbol{\psi}^-$, then the linear combination for it cannot involve $\boldsymbol{\psi}^+$ because

† We can find such a weight on F that $\{\psi_n^+(\lambda, t)|n$=0, 1, 2, ...$\}$ forms a system of (normalized) orthogonal polynomials. Then $\{g_0(\lambda, t)\psi_n^-(\lambda, t)|n$=0, 1, 2, ...$\}$ is a system of the second-kind functions, $g_0\psi_n^-$ being related to ψ_n^+ by the Stieltjes transform (see Yoshino (1987)).

of the behaviours at $\lambda \to \infty$ for $n \gg 1$. We thus obtain by using a suitable function $f(\lambda, t)$ (analytic with respect to λ)

$$\partial_t \psi^-(\lambda, t) - \mathbf{B}(t)\psi^-(\lambda, t) = f(\lambda, t)\psi^-(\lambda, t)$$

which leads to the equation of motion for $g_n(\lambda, t)$ $(= \psi_n^-/a_{n-1}\psi_{n-1}^-)$:

$$\partial_t[g_n(\lambda; t)] = 1 - (\lambda - b_n)g_n(\lambda; t) + a_{n-1}^2 g_n^2(\lambda; t) \tag{4.15}$$

after slightly tedious calculations using (3.3), (3.4), (4.2) and (4.14).

We determine the dependence of σ by letting $\lambda \to \zeta_n$ in (4.15). We obtain (see (4.4) and (4.10), and note that ζ_n is a zero of $g_n(\lambda; t)$)

$$\partial_t \sigma = 2\partial_t[u(\zeta_n)] = \gamma(F)\frac{2\vartheta_1(\frac{1}{2}\kappa)}{\vartheta_1'} \tag{4.16}$$

because equations (4.8) and (4.9), with (4.4) substituted, lead to

$$g_n(\lambda, t) = \frac{-\vartheta_1'\Delta u}{\gamma(F)\vartheta_1(\frac{1}{2}\kappa)} + \mathrm{O}(\Delta u^2) \qquad \Delta u = \Delta u(\lambda, t) \equiv u(\lambda) - u(\zeta_n)$$

for $\lambda \to \zeta_n$. Evidently, the quantity given by (4.16) is nothing but the angular frequency. Equation (4.12) then becomes

$$v_n = n\kappa(F) - \gamma(F)\frac{2\vartheta_1(\frac{1}{2}\kappa)}{\vartheta_1'}t + \delta \tag{4.17}$$

where δ is an integration constant. Equation (4.11) now becomes complete as an expression for $a_n = a_n(t)$.

4.3. Expression for y_n

Let us express y_n. Equation (4.11) is substantially the same as (2.3). In the present case, unlike in section 2, we want to have y_n expressed in the form of a function of F. To do so the second of (3.9) is useful (see the argument in section 3.3), and we obtain

$$y_n = A + \Lambda(F)t - 2n\ln\gamma(F) + \ln\left[\frac{\vartheta_4(\frac{1}{2}v_{n-1})}{\vartheta_4(\frac{1}{2}v_n)}\right] \tag{4.18}$$

where A is an integration constant. In the solution for y_n, i.e. (4.18) accompanied with (4.17), we find four parameters unless we take the integration constants, A and δ, into account (see remark (i) in section 2). Two, $\gamma(F)$ and $\Lambda(F)$, are quantitative, and the other two, $\kappa(\Gamma)$ and $q(\Gamma)$, are qualitative, in accordance with the discussion in section 3.1.

4.4. Explicit relations between the parameters and the spectrum

The four parameters in a set of (4.18) and (4.17) are functions of F as we observed in section 3. In this section we give explicit relations between them in terms of the theta functions (see also Yoshino (1988)). We first relate $q(\Gamma)$ and $\kappa(\Gamma)$ to F. Instead of the former we give an expression for the corresponding modulus k. Then we express the other parameters by using λ_1, λ_2, λ_3, λ_4 and the theta functions (with nome q) whose arguments are $\frac{1}{2}\kappa$ or 0. In the following we use the notations:

$$\rho_1 = \rho_1(\Gamma) \equiv \frac{\vartheta_1'(\frac{1}{2}\kappa)}{\vartheta_1'} \qquad \rho_j = \rho_j(\Gamma) \equiv \frac{\vartheta_j(\frac{1}{2}\kappa)}{\vartheta_j} \qquad (j = 2, 3, 4)$$

for simplicity and clearness. They are all qualitative.

Let us consider a function (of λ): $(\lambda-\lambda_m)/(\lambda_l-\lambda_m)$ $(l, m = 1, 2, 3, 4;\ l \neq m)$, which is algebraic on $\mathcal{S}$. It has a double zero at λ_m and two simple poles at ∞ and ∞^*, and becomes unity when $\lambda = \lambda_l$. Therefore we obtain

$$\frac{\lambda-\lambda_m}{\lambda_l-\lambda_m} = \frac{\vartheta_1(u(\lambda_l)-u(\infty))\vartheta_1(u(\lambda_l)-u(\infty^*))\vartheta_1^2(u(\lambda)-u(\lambda_m))}{\vartheta_1^2(u(\lambda_l)-u(\lambda_m))\vartheta_1(u(\lambda)-u(\infty))\vartheta_1(u(\lambda)-u(\infty^*))}. \tag{4.19}$$

We introduce the notation: $[i, j/l, m] \equiv (\lambda_i-\lambda_j)/(\lambda_l-\lambda_m)$ for brevity. Then equation (4.19) gives rise to (see figure 2(b))

$$[1,3/1,4] = \frac{\vartheta_3^2\vartheta_1^2(\frac14\kappa)}{\vartheta_2^2\vartheta_4^2(\frac14\kappa)} \qquad [2,3/2,4] = \frac{\vartheta_2^2\vartheta_1^2(\frac14\kappa)}{\vartheta_3^2\vartheta_4^2(\frac14\kappa)} \qquad [2,4/1,4] = \frac{\vartheta_3^2\vartheta_2^2(\frac14\kappa)}{\vartheta_2^2\vartheta_3^2(\frac14\kappa)}.$$

We thus obtain

$$k^2 = \frac{\vartheta_2^4}{\vartheta_3^4} = \frac{[2,3/2,4]}{[1,3/1,4]} = \frac{(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)} \tag{4.20}$$

$$\begin{aligned}\mathrm{cn}\left(\frac{K\kappa}{\pi}\right) &= \frac{\rho_2}{\rho_4} = \frac{\vartheta_2^2(\frac14\kappa)\vartheta_4^2(\frac14\kappa) - \vartheta_1^2(\frac14\kappa)\vartheta_3^2(\frac14\kappa)}{\vartheta_2^2(\frac14\kappa)\vartheta_4^2(\frac14\kappa) + \vartheta_1^2(\frac14\kappa)\vartheta_3^2(\frac14\kappa)} \\ &= \frac{[2,4/1,4]-[1,3/1,4]}{[2,4/1,4]+[1,3/1,4]} = \frac{-\lambda_1+\lambda_2+\lambda_3-\lambda_4}{\lambda_1+\lambda_2-\lambda_3-\lambda_4}.\end{aligned} \tag{4.21}$$

In (4.21) we have used the duplication formulae (WW, p 488). Equation (4.21) determines κ uniquely when F or Γ is given, because κ ranges between 0 and 2π over which $\mathrm{cn}(K\kappa/\pi)$ is monotone (decreasing). In a similar manner as (4.21) we obtain (note $[1,4/2,4] = [2,4/1,4]^{-1}$)

$$\mathrm{dn}\left(\frac{K\kappa}{\pi}\right) = \frac{\rho_3}{\rho_4} = \frac{[1,4/2,4]-[2,3/2,4]}{[1,4/2,4]+[2,3/2,4]} = \frac{\lambda_1-\lambda_2+\lambda_3-\lambda_4}{\lambda_1+\lambda_2-\lambda_3-\lambda_4} \tag{4.22}$$

which has a similar form to (4.21), but cannot play as a perfect substitute for (4.21) because $\mathrm{dn}(K\kappa/\pi)$ is not monotone in $0 < \kappa < 2\pi$.

In order to express $\gamma(F)$ we consider a function: $(\lambda-\lambda_m)\Phi(\lambda; F)$. Its logarithmic differential is of the third kind with poles at λ_m (residue $= 2$) and ∞ (residue $= -2$), and it takes the value $\gamma(F)^2$ when $\lambda \to \infty^*$ (see (4.6)). We thus obtain

$$(\lambda-\lambda_m)\Phi(\lambda; F) = \gamma(F)^2 \frac{\vartheta_1^2(\frac12\kappa)\vartheta_1^2(u(\lambda)-u(\lambda_m))}{\vartheta_1^2(u(\lambda_m)-u(\infty^*))\vartheta_1^2(u(\lambda)-u(\infty))}$$

which gives rise to

$$\lambda_1-\lambda_3 = 4\gamma(F)\frac{\vartheta_1^2(\frac14\kappa)\vartheta_3^2(\frac14\kappa)}{\vartheta_2^2\vartheta_4^2} \qquad \lambda_2-\lambda_4 = 4\gamma(F)\frac{\vartheta_2^2(\frac14\kappa)\vartheta_4^2(\frac14\kappa)}{\vartheta_2^2\vartheta_4^2}$$

where we have applied the duplication formula to $\vartheta_1(\frac12\kappa)$. It is then straightforward to have

$$\gamma(F) = \frac{\lambda_1+\lambda_2-\lambda_3-\lambda_4}{4\rho_4} \tag{4.23}$$

by using the duplication formula for $\vartheta_4(\frac12\kappa)$. We can derive a similar expression that makes reference to ρ_2 or ρ_3 instead of ρ_4, by substituting (4.21) or (4.22) into (4.23).

Finally we settle $\Lambda(F)$ and at the same time $\omega(F)$. Let us expand expression (4.8) around ∞. The l.h.s. becomes (3.7). We can readily expand the r.h.s. by using the expansions (use (3.12) for the first one):

$$\Delta u \equiv u(\lambda) - u(\infty) = \tfrac{1}{2}\omega(F)[\lambda^{-1} + \tfrac{1}{4}(\lambda_1+\lambda_2+\lambda_3+\lambda_4)\lambda^{-2} + \mathrm{O}(\lambda^{-3})]$$
$$\vartheta_1(u(\lambda) - u(\infty)) = \vartheta_1' \Delta u + \mathrm{O}(\Delta u^3)$$
$$\vartheta_1(u(\lambda) - u(\infty^*)) = \vartheta_1(\tfrac{1}{2}\kappa) + \vartheta_1'(\tfrac{1}{2}\kappa)\Delta u + \mathrm{O}(\Delta u^2).$$

Then we obtain by comparing the terms of λ^1 and λ^0 (note $\omega(F)/\gamma(F)$ is qualitative as we observed in section 3.5)

$$\omega(F) = \gamma(F)\frac{2\vartheta_1(\frac{1}{2}\kappa)}{\vartheta_1'} \tag{4.24}$$

$$\Lambda(F) = \tfrac{1}{4}(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) - \gamma(F)\rho_1. \tag{4.25}$$

Comparing (4.24) with (4.17), we find that $\omega(F)$ (defined by (3.12)) equals the angular frequency of the periodic wave, as expected in section 3.5. We note that equation (4.24) is substantially the same as the dispersion relation (2.7) in section 2. Substituting (4.23) into (4.24) we obtain an alternative expression for $\omega(F)$:

$$\omega(F) = \frac{\pi}{4K}(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)\mathrm{sn}\left(\frac{K\kappa}{\pi}\right)$$

which is written by using a more familiar (Jacobian) function, but is less fit for solution (4.18) that makes no direct reference to λ_1, λ_2, λ_3 or λ_4.

To make the relations clearer express F, i.e. λ_1, λ_2, λ_3 and λ_4, as functions of the four parameters, $\gamma(F)$, $\Lambda(F)$, $\kappa(\Gamma)$ and $q(\Gamma)$. From equations (4.21)–(4.23) and (4.25) we obtain

$$\begin{aligned}
\lambda_1 &= \Lambda(F) + \gamma(F)(\rho_1 - \rho_2 + \rho_3 + \rho_4)\\
\lambda_2 &= \Lambda(F) + \gamma(F)(\rho_1 + \rho_2 - \rho_3 + \rho_4)\\
\lambda_3 &= \Lambda(F) + \gamma(F)(\rho_1 + \rho_2 + \rho_3 - \rho_4)\\
\lambda_4 &= \Lambda(F) + \gamma(F)(\rho_1 - \rho_2 - \rho_3 - \rho_4)
\end{aligned} \tag{4.26}$$

where we should note that such an expression as $\lambda_1 = \lambda_1(\gamma, \Lambda, \kappa, q) = \Lambda + \gamma[\rho_1(\kappa, q) - \cdots]$ etc is rather sound. Equations (4.26) manifest the (quantitative) roles of $\gamma(F)$ and $\Lambda(F)$ again.

In section 2 we referred to (2.7), which is equivalent to (4.24), as the dispersion relation, i.e. we regarded κ as varying independently of the other parameters. It might then be helpful to know how F varies in this picture. For this purpose we give figure 4, which is easily obtained by (4.26).

4.5. *Time reversal*

Equation (1.1), i.e. the original equation of motion, bears time reversal symmetry. Namely, if $y_n(t)$ is a solution, then $y_n(-t)$ must also be a solution. However, our solution, i.e. a set of (4.18) and (4.17), does not explicitly exhibit this property because we did not conserve this symmetry in the Lax-form variables (see (3.1) and (3.4)). In the present treatment the reversal of time corresponds to the reversal of the spectrum, which we show in the following (see also remark (ii) in section 2).

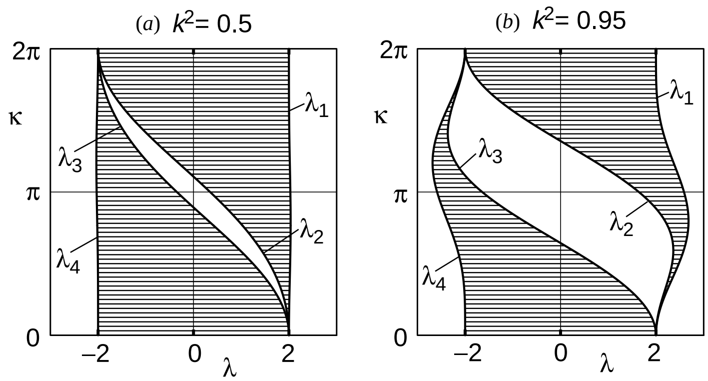

Figure 4. The spectrum F versus κ with k^2 (i.e. q) fixed. The quantitative parameters are kept as $\gamma(F) = 1$ and $\Lambda(F) = 0$, i.e. with no expansion (or shrinkage) nor translational motion.

By the superscript 'r', we mean a reversal of F, i.e.

$$F^{\mathrm{r}} = \{\lambda | -\lambda \in F\} = [\lambda_4^{\mathrm{r}}, \lambda_3^{\mathrm{r}}] + [\lambda_2^{\mathrm{r}}, \lambda_1^{\mathrm{r}}]$$
$$\lambda_m^{\mathrm{r}} = -\lambda_{5-m} \qquad (m = 1, 2, 3, 4)$$

and Γ^{r} is the corresponding class. Note that λ_m^{r} are chosen to be in descending order. We obtain

$$\gamma(F^{\mathrm{r}}) = \gamma(F) \qquad \Lambda(F^{\mathrm{r}}) = -\Lambda(F) \tag{4.27}$$

because we readily find $\Phi(\lambda; F^{\mathrm{r}}) = -\Phi(-\lambda; F)$ by definition (see section 3.3). We also obtain

$$q(\Gamma^{\mathrm{r}}) = q(\Gamma) \qquad \kappa(\Gamma^{\mathrm{r}}) = 2\pi - \kappa(\Gamma) \qquad \omega(F^{\mathrm{r}}) = \omega(F) \tag{4.28}$$

from (4.20), (4.21) and (4.24), respectively. We note that the first two of (4.28) directly give

$$\rho_1(\Gamma^{\mathrm{r}}) = -\rho_1(\Gamma) \qquad \rho_2(\Gamma^{\mathrm{r}}) = -\rho_2(\Gamma) \qquad \rho_3(\Gamma^{\mathrm{r}}) = \rho_3(\Gamma) \qquad \rho_4(\Gamma^{\mathrm{r}}) = \rho_4(\Gamma)$$

which without a doubt confirm (4.23), (4.25) and (4.26) for F^{r}.

Now let us denote expression (4.18) (along with (4.17)) by $y_n(t; F, A, \delta)$ with *all* parameters specified. Then we find

$$y_n(-t; F, A, \delta) = y_n(t; F^{\mathrm{r}}, A, -\delta) \tag{4.29}$$

by using (4.27) and (4.28). Namely, $y_n(-t)$ surely becomes a solution but it corresponds to the reversed spectrum.

4.6. Multigap cases

Until now we have dealt with a single-gap spectrum. We nevertheless note that the procedure we have presented is applicable with some modifications to the multigap spectrum cases. In the following we briefly refer to them. We will report the details elsewhere.

The arguments in sections 3 and 4.1–4.3 require only slight modifications. Let p be the number of gaps, i.e. if there is $2p + 2$ parameters (bandedge points), then the relevant Riemann surface is of genus p. The classification of spectra in section 3.1 is evidently valid and there should be $2p$ qualitative parameters. In the p-gap case $\mathrm{d}u(\lambda; F)$, $u(\lambda; F)$, $\kappa(\Gamma)$ and $\omega(F)$ become p-dimensional vectors, and $\tau(\Gamma)$ becomes p-dimensional symmetric matrix, which introduces the Riemann theta functions (of p variables) instead of the elliptic theta functions.

To view of the general properties of $\Phi(\lambda; F)$ and $g_n(\lambda; F)$ we need no modifications (inclusive of the equation of motion (4.15)), which both remain scalar functions (see Yoshino 1987). Two quantitative parameters, $\gamma(F)$ and $\Lambda(F)$, are thus defined in the same manner, and expansions (4.7) remain valid. The number of the zeros and that of the poles of $g_n(\lambda; F)$, not counting ∞ (zero) nor ∞^* (pole), both become p with exactly one zero and one pole being in each gap (the sheets they belong to being indefinite). These facts make (4.18), if $\vartheta_4(\cdot)$ is replaced by an adequate Riemann theta function, be a solution also in the p-gap case. Equation (4.24) (or (4.16)) becomes a form of linear simultaneous equations for p elements of $\omega(F)$ (or $\partial_t\sigma$).

More modifications are necessary for the argument in section 4.4. One cause is that we know far less formulae in view of the Riemann theta functions than the elliptic theta functions, for example those corresponding to the duplication formulae we often used. One problem we should solve is what we should choose as $2p$ qualitative parameters.

5. One-soliton limit

We derive a one-soliton solution from the simply periodic solution by taking a limit. We use the expression in section 4. The limit we should take is letting either subband shrink to a point, i.e. $\lambda_1 \to \lambda_2 + 0$ (case 1) or $\lambda_4 \to \lambda_3 - 0$ (case 2). In the following we treat case 1. Case 2 is substantially the same as case 1 because we can relate these two cases to each other by reversing the spectrum (see section 4.5).

We take the limit by keeping both $\gamma(F)$ and $\Lambda(F)$ fixed. This is always possible because we can pick a spectrum F out of any class such that $\gamma(F)$ and $\Lambda(F)$ equal (arbitrarily) given values. We thus begin with an estimation of the two qualitative parameters. We obtain $k \to 1-0$ from (4.20), which implies $q \to 1-0$ and $\tau \to +\mathrm{i}0$. The other parameter κ then becomes $\kappa = \mathrm{O}(K(k)^{-1}) \to +0$ from (4.21).

Because $q \to 1 - 0$, we should apply Jacobi's imaginary transformation (WW, p 474), which gives rise to the theta functions with the complementary nome q' $(\to +0)$. In the following we specify the nomes in the arguments. The estimate for κ implies κ/τ is kept finite in the limit. We therefore let $\xi = \mathrm{i}\kappa/2\tau$ $(\in \mathbb{R}, > 0)$, and then take the limit $q' \to +0$ keeping ξ fixed (in addition to $\gamma(F)$ and $\Lambda(F)$). The qualitative parameters are now q' and ξ, and there remains only ξ when the limit is taken.

Relation (4.24) reads as

$$\frac{\omega}{\kappa} = -\mathrm{i}\gamma(F)\frac{\exp(\xi\kappa/2\pi)\vartheta_1(\mathrm{i}\xi, q')}{\xi\vartheta_1'(0, q')} \to \gamma(F)\frac{\sinh\xi}{\xi}$$

because $\vartheta_1(z, q') = 2q'^{1/4}\sin z + \mathrm{O}(q'^{9/4})$ (WW, p 464). We thus rewrite (4.17) as

$$\frac{\mathrm{i}v_n}{2\tau} = (n\kappa - \omega t)\frac{\xi}{\kappa} + \delta \to n\xi - t\gamma(F)\sinh\xi + \delta \equiv w_n \qquad (\in \mathbb{R}) \tag{5.1}$$

where we have redefined an arbitrary phase factor δ. We should regard w_n as finite, which implies $v_n \to 0$. The significant part of solution (4.18) is hence written as

$$\frac{\vartheta_4(\frac{1}{2}v_{n-1}, q)}{\vartheta_4(\frac{1}{2}v_n, q)} = \exp\left[\frac{\xi}{2\pi}(v_{n-1}+v_n)\right]\frac{\vartheta_2(v_{n-1}/2\tau, q')}{\vartheta_2(v_n/2\tau, q')} \to \frac{\cosh w_{n-1}}{\cosh w_n}$$

because $\vartheta_2(z, q') = 2q'^{1/4}\cos z + \mathrm{O}(q'^{9/4})$. Equation (4.18) then becomes

$$y_n = A + \Lambda(F)t - 2n\ln\gamma(F) + \ln\left(\frac{\cosh w_{n-1}}{\cosh w_n}\right) \tag{5.2}$$

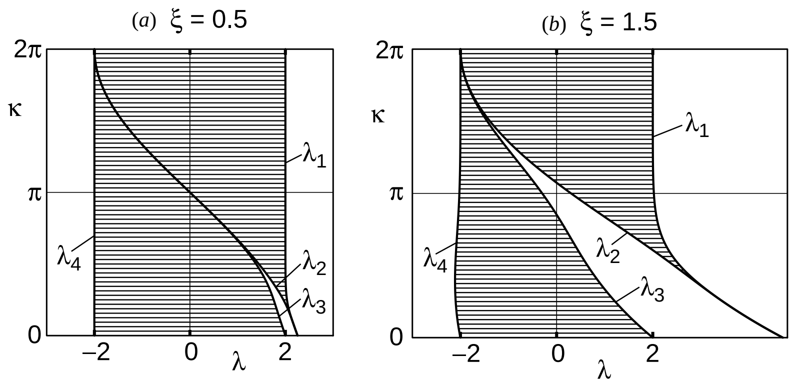

Figure 5. The spectrum F versus κ with ξ fixed. The quantitative parameters are kept as $\gamma(F) = 1$ and $\Lambda(F) = 0$. The soliton limit is $\kappa \to +0$.

which is written by using three (significant) parameters: $\gamma(F)$, $\Lambda(F)$ and ξ. The last term in (5.2) expresses a soliton, whose velocity is, we note, proportional to $\gamma(F)$.

We here impose the boundary condition. As equation (5.2) suggests, it should be written as $|y_n| < \infty$ and $\mathrm{d}y_n/\mathrm{d}t \to 0$ for $n \to \pm\infty$. This gives rise to the same result as before, i.e. the quantitative parameters come to be definite as $\gamma(F) = 1$ and $\Lambda(F) = 0$. We cannot impose $y_n \to 0$ at *both* of $\pm\infty$ while it is possible to impose $r_n \to 0$, as is well known. We have successfully obtained one-soliton solution (see Toda (1981)). The bandedge points, given by (4.26), now become

$$\lambda_1, \lambda_2 \to 2\cosh\xi \qquad \lambda_3 \to 2 \qquad \lambda_4 \to -2$$

because $\rho_1, \rho_4 \to \cosh\xi$ and $\rho_2, \rho_3 \to 1$. Figure 5 is given to help the reader understand the limit intuitively.

It is easy to take case 2 into consideration. Equation (4.29) states that we can obtain the solution corresponding to F^{r} (i.e. in case 2) by replacing $t \to -t$ in the solution corresponding to F (i.e. in case 1). Thus, we can regard (5.2) as inclusive of both cases 1 and 2, if we read w_n as $w_n \equiv n\xi \pm t\gamma(F)\sinh\xi + \delta$ instead of (5.1) (note the second term in (5.2) can be readily matched, if necessary, by the translation of F). Evidently a mirror reflection (about $\lambda = 0$) of figure 5 becomes the case 2 version of figure 5.

6. Concluding remarks

In this paper we have been concerned with a simply periodic solution in the Toda lattice. By using the identities of the elliptic theta functions we first (in section 2) showed that equation (2.1) accompanied with (2.7), not with (1.4), represents a simply periodic solution. We next made the solution clearer by deriving and discussing it from the viewpoint of the inverse spectrum problem. We note relation (4.24), which is essentially equivalent to (2.7), was obtained in due course of the derivation. We also note that two functions, $g_n(\lambda; F)$ and $\Phi(\lambda; F)$, played essential roles.

We finally noted that the Toda lattice has not yet been fully understood. In this paper we referred to a simply periodic solution (and one-soliton solution as the limit) only. It seems that there are many problems left indefinite in the Toda lattice problem.

References

Date E and Tanaka S 1976a *Prog. Theor. Phys.* **55** 457–65
——1976b *Prog. Theor. Phys. Suppl.* **59** 107–25
Flaschka H 1974 *Phys. Rev.* B **9** 1924–5
Hille E 1962 *Analytic Function Theory* vol 2 (New York: Ginn) ch 16
Kac M and van Moerbeke P 1975a *Proc. Natl Acad. Sci., USA* **72** 1627–9
——1975b *Proc. Natl Acad. Sci., USA* **72** 2879–80
Magnus A 1979 *Padé Approximation and its Applications (Lecture Notes in Mathematics 765)* ed L Wuytack (Berlin: Springer) pp 150–71
Peherstorfer F 1991 *J. Approx. Theor.* **64** 123–61
Siegel C L 1969 *Topics in Complex Functions* vol 1 (New York: Wiley)
——1971 *Topics in Complex Functions* vol 2 (New York: Wiley) ch 4
Toda M 1967a *J. Phys. Soc. Japan* **22** 431–6
——1967b *J. Phys. Soc. Japan* **23** 501–6
——1981 *Theory of Nonlinear Lattices (Springer Series in Sold-State Sciences 20)* (Berlin: Springer)
Turchi P, Ducastelle F and Tréglia G 1982 *J. Phys. C: Solid State Phys.* **15** 2891–924
van Moerbeke P 1976 *Inv. Math.* **37** 45–81
Whittaker E T and Watson G N 1927 *A Course of Modern Analysis* 4th edn (Cambridge: Cambridge University Press)
Yoshino S 1987 *J. Phys. A: Math. Gen.* **20** 4129–52
——1988 *J. Phys. A: Math. Gen.* **21** 1533–47